Handling Infinitely Branching WSTS

Michael Blondin [1 2], Alain Finkel[1] & Pierre McKenzie [1 2]

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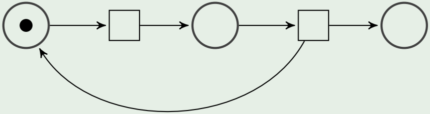
PV 2015, Madrid, September 4, 2015

Well-structured transition systems (WSTS) encompass a large number of infinite state systems.

Example of WSTS: Petri nets (Geeraerts, Heußner, Praveen & Raskin PN'13)



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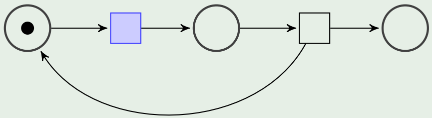

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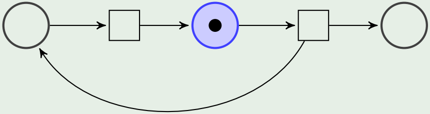

Multiple decidability results are known for finitely branching WSTS.

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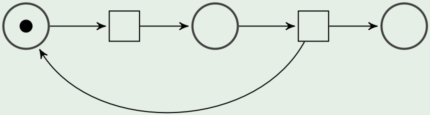

$$\mathrm{Post}(\odot \; \bigcirc \; \bigcirc) \;=\; \bigcirc \; \odot \; \bigcirc$$

How to handle infinitely branching WSTS such as systems with infinitely many initial states, and parametric systems?

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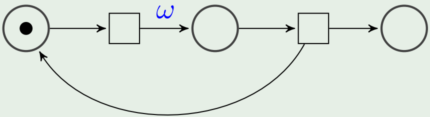

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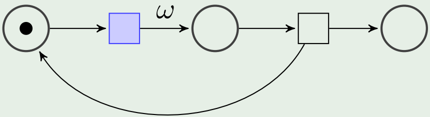

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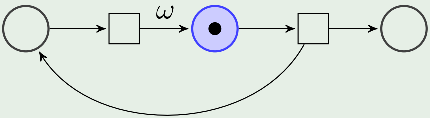

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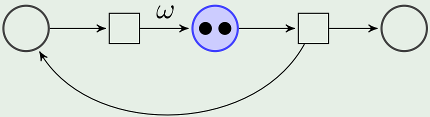

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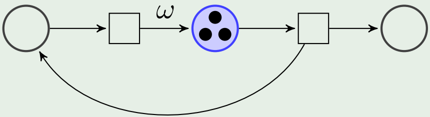

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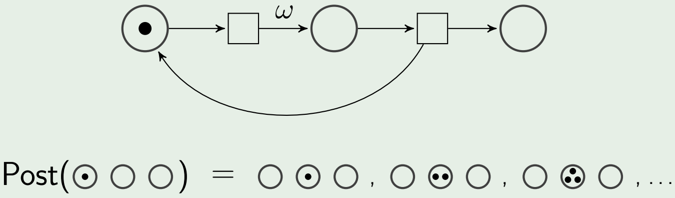
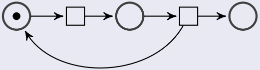


Well-structured transition system (Finkel ICALP'87, Finkel & Schnoebelen TCS'01)

$S = (X, \rightarrow, \leq)$ where

- X set,
- $\rightarrow \subseteq X \times X$,
- monotony,
- well-quasi-ordered.



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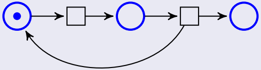

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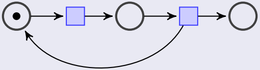

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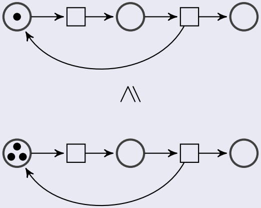

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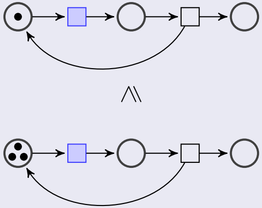

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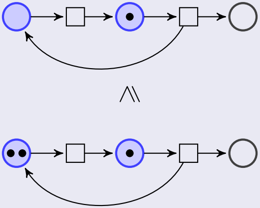

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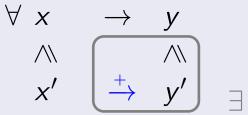

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  $\forall x_0, x_1, \ldots \ \exists i < j$ s.t. $x_i \leq x_j$.

Branching

A WSTS $(X, \to, \leq)$ is *finitely branching* if $\text{Post}(x)$ is finite for every $x \in X$.

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- Much more.

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- ω-Petri nets (Geeraerts, Heussner, Praveen & Raskin PN'13),
- Parametric WSTS.

Proposition

Finite branching is undecidable for post-effective WSTS.

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Proof

- Let $S_i = (\mathbb{N}, \rightarrow_{S_i}, \leq)$ be the WSTS such that:
  - $x \rightarrow_{S_i} x + 1$ if TM_i does not halt within $\leq x$ steps,
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- S_i has strong and strict monotony since $x \rightarrow_{S_i} x + 1$ for every $x \in \mathbb{N}$.

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- S_i has strong and strict monotony since $x \rightarrow_{S_i} x + 1$ for every $x \in \mathbb{N}$.
- TM_i halts iff there exist $x \in \mathbb{N}$ and an execution $0 \xrightarrow{*}_{S_i} x$ such that $\text{Post}_{S_i}(x)$ is infinite.
- The halting problem thus Turing-reduces to the infinite branching problem.

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We want to study the usual reachability problems for these infinitely branching systems, e.g.,

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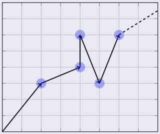

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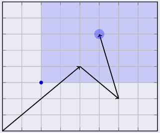

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We want to study the usual reachability problems for these infinitely branching systems, e.g.,

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- Coverability,
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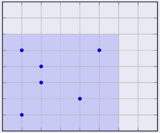
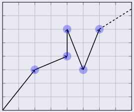


Termination

Input: $(X, \to, \leq)$ a WSTS, $x_0 \in X$.

Question: $\exists x_0 \to x_1 \to x_2 \to \ldots$?



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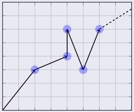

Theorem (Finkel ICALP'87)

Termination is decidable for finitely branching WSTS with transitive monotony.

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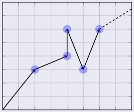

Theorem (deduced from Dufourd, Jančar & Schnoebelen ICALP'99)

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Remark

Strong termination and termination are the same in finitely branching WSTS.

Strong termination

Input: $(X, \rightarrow, \leq)$ a WSTS, $x_0 \in X$.

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Theorem

Strong termination is decidable for infinitely branching WSTS under some assumptions.

Issues with finite branching techniques

Some techniques for WSTS based on finite reachability trees; impossible for infinite branching.

Some rely on upward closed sets; what about downward closed, in particular with infinite branching?

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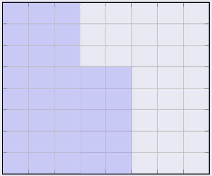
A tool

Develop from the WSTS *completion* introduced by Finkel & Goubault-Larrecq in STACS'09 and ICALP'09.

Ideals

$I \subseteq X$ is an *ideal* if

- downward closed: $I = \downarrow I$,



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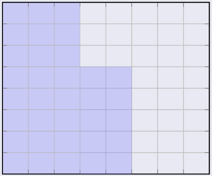

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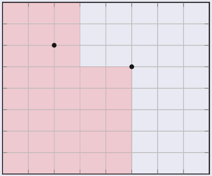

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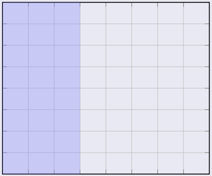
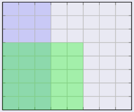


Theorem (Finkel & Goubault-Larrecq ICALP'09; Goubault-Larrecq '14)

$$D \text{ downward closed } \implies D = \bigcup_{\text{finite}} \text{Ideals}$$



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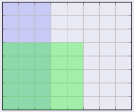

Corollary

Every downward closed set decomposes canonically as the union of its maximal ideals.

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- $\widehat{X} = \mathrm{Ideals}(X)$,
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Theorem

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Let S be a WSTS, then $\widehat{S}$ is a WSTS iff S is a ω^2-WSTS.

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Relating executions of S and $\widehat{S}$

Let $S = (X, \rightarrow_S, \leq)$ be a WSTS with transitive monotony, then

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Relating executions of S and $\widehat{S}$

Let $S = (X, \to_S, \leq)$ be a WSTS with strong monotony, then

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Relations between S and $\widehat{S}$

A generality

The completion $\widehat{S} = (\widehat{X}, \rightarrow_{\widehat{S}}, \subseteq)$ computes exactly the downward closure of the reachability set of its original system $S = (X, \rightarrow_S, \leq)$.

An equality

We have: $\mathrm{Post}^*_{\widehat{S}}(\downarrow x) = \downarrow \mathrm{Post}^*_S(x)$.

In fact, it is more exactly:

Theorem

If $Post^*_{\widehat{S}}(\downarrow x) = \{J_1, \ldots, J_n\}$ *then* $\downarrow Post^*_S(x) = J_1 \cup \ldots \cup J_n$.

Theorem

Strong termination is decidable for infinitely branching WSTS with transitive monotony and such that $\widehat{S}$ is a post-effective WSTS.

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Post-effectiveness

Possible to compute cardinality of

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- Executions bounded in S iff bounded in $\widehat{S}$.

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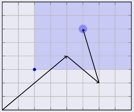
Proof

- Executions bounded in S iff bounded in $\widehat{S}$.
- $\widehat{S}$ finitely branching, can decide termination in $\widehat{S}$ by Finkel ICALP'87, Finkel & Schnoebelen TCS'01.

Coverability

Input: $(X, \rightarrow, \leq)$ a WSTS, $x_0, x \in X$.

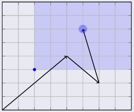
Question: $x_0 \xrightarrow{*} x' \geq x$?



Coverability

Input: $(X, \to, \leq)$ a WSTS, $x_0, x \in X$.

Question: $x_0 \in \uparrow \mathrm{Pre}^*(\uparrow x)$?



Backward method (Abdulla, Cerans, Jonsson & Tsay IC'00)

Compute $\uparrow \mathrm{Pre}^*(\uparrow x)$ iteratively assuming $\uparrow \mathrm{Pre}(\uparrow x)$ computable.

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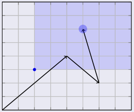

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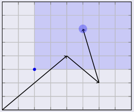

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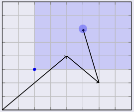

Forward method

Coverability:

- Enumerate executions $\downarrow x_0 \xrightarrow{*}_{\widehat{S}} I$,
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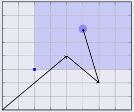

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- Enumerate executions $\downarrow x_0 \xrightarrow{*}_{\widehat{S}} I$,
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Non coverability:

- Enumerate $D \subseteq X$ downward closed, $x_0 \in D$ and $\downarrow \text{Post}_S(D) \subseteq D$
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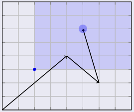

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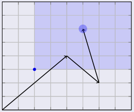

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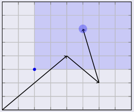

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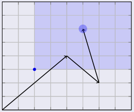

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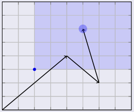

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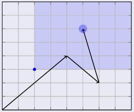

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Prebasis is not computable for $\mathcal{F}_1$

Let $S_i = (\mathbb{N}, \rightarrow_{S_i}, \leq)$ be the WSTS such that:

- $x \rightarrow_{S_i} 0$ if TM_i does not halt on its encoding in $\leq x$ steps,
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Then $S_i \in \mathcal{F}_1$ and S_i is effective.

Three Pre sets

- $Pre_{S_i}(0) = \{x \in \mathbb{N} : TM_i$ does not halt in $\leq x$ steps $\}$,
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- But $\mathrm{Pre}_{S_i}(1) = \emptyset$ iff TM_i does not halt.
- The halting problem thus Turing-reduces to the prebasis computation.

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- If c has an ancestor c' labelled x' such that $x' < x$, we return "unbounded". Otherwise,
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  - Otherwise, if $\text{Post}_S(x)$ is infinite, then we return "unbounded". Otherwise we mark c and for every $y \in \text{Post}_S(x)$ we add a child labelled y to c.

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- T is finite and correct.

Further result for infinitely branching WSTS

Strong maintainability is decidable for WSTS with strong monotony and such that $\widehat{S}$ is a post-effective WSTS.

Further work

- ∃ general class of infinitely branching WSTS with a Karp-Miller procedure?

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- Toward the algorithmics of complete WSTS.
- What else can we do with the WSTS completion?

Thank you!